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# Anderson Parabolic Model for a Quasi-Stationary Medium

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**Abstract** We study the Anderson Parabolic Model for a random medium which is a product of an i.i.d. space-like random field and a white noise. The model has long range space-time correlations and is intermediate between the stationary case and the "turbulent" one, which were studied in previous works. Under some natural assumptions on the distribution of the space potential, we prove existence and uniqueness, and derive the long time asymptotics for the annealed moments, and the "semi-annealed" ones, for which expectation is taken only w.r.t. the white noise. A conjecture for the fully quenched case is discussed on a simplified model.

## 1 Introduction. Statement of the results

The Anderson parabolic problem on the lattice $\mathbb{Z}^d$, $d \geq 1$, has the general form

$$\frac{\partial u}{\partial t} = Hu := \kappa \Delta u + V(t, x, \omega)u, \qquad u(0, x) \equiv 1, \quad x \in \mathbb{Z}^d,\ t \in \mathbb{R}_+, \tag{1.1}$$

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where $H = \kappa\Delta + V$ is a random Schrödinger operator, $\Delta$ is the lattice Laplacian

$$\Delta\psi = \frac{1}{2d} \sum_{x':|x-x'|=1} (\psi(x') - \psi(x)) \tag{1.2}$$

and $V$ is a random potential, which can be stationary: $V(t,x,\omega) = V(x,\omega)$. $\omega \in \Omega$ denotes a particular realization of the random medium $\{V(\cdot)\}$, which is homogeneous and ergodic in space-time, or in space for the stationary case.

The stationary case was studied in several papers by Gärtner and Molchanov (see for instance the basic publications [1, 2]). Non-stationary potentials, which are usually associated to the theory of magnetic fields in random flows (see [3]), under the assumption that correlations are short-range in space and time, were the subject of the memoir [4]. For a general discussion of the theory of random media and of their qualitative properties (such as intermittency, localization etc.), see [5].

Space-time uncorrelated random media appear in many models, such as the discrete time random walks in the recent paper [6].

In the present paper we consider the quasi-stationary case, for which

$$V(t,x) = \dot{w}_t V(x) \tag{1.3}$$

where $\dot{w}_t$ is the standard Gaussian white noise, i.e., the derivative of the standard Wiener process, and $\{V(x) : x \in \mathbb{Z}^d\}$ are i.i.d. r.v.'s, independent of the white noise. Such a random medium has, of course, long-range space-time correlations.

The idea of a quasi-stationary medium is due to K. Khanin, who proposed it for the study of the growth of a random polymer in a random environment. Our model can be seen as something intermediate between the time-stationary case $V = V(x,\omega)$ and the "turbulent" case $V = V(x,\omega)\dot{w}_t(x)$, where $\{\dot{w}_t(x), x \in \mathbb{Z}^d\}$ are independent copies of the standard white noise, independent of $V(x,\omega)$.

We assume that the random medium distribution is given by an independent pair: if $(\Omega_w, \mathcal{F}_w, P_w)$ and $(\Omega_V, \mathcal{F}_V, P_V)$ are the probability spaces of the white noise and the random potential, then our probability space $(\Omega, \mathcal{F}, P)$ is a product: $\Omega = \Omega_w \times \Omega_V$, where $\omega_w = \{w_\cdot\}$, and $\omega_V = \{V(\cdot)\}$, and $\mathcal{F} = \mathcal{F}_w \times \mathcal{F}_V$, $P = P_w \times P_V$.

Expectations which refer to the environment are usually denoted as $\langle\cdot\rangle$, with possible pedices if the probability distribution is not $P = P_w \times P_V$.

It is usual to assume a Gaussian distribution for the potential, but this is not appropriate for our case, since the annealed moments would not exist (see remark at the end of Sect. 4). We deal instead with Weibull distributions (see, e.g. [9]). The random variable $\xi$ is said to have a Weibull distribution with parameters $\alpha > 1$ and $c > 0$ if, for all $a > 0$

$$P(\{\xi \geq a\}) = e^{-c\frac{a^\alpha}{\alpha}}. \tag{1.4}$$

For the purposes of the present paper it would be natural to assume that the quantities $V^2(x)$ have a distribution with regular tails. This is a more general class of distributions, for which (1.4) is replaced by $P(\{\xi \geq a\}) = \exp\{-\frac{a^\alpha}{\alpha}L(a)\}$, for some slowly varying (in the sense of Karamata) function $L$. Under mild technical assumption on $L(a)$ ("normality", i.e., $L'(a) = o(L(a)/a)$ as $a \to \infty$), using the results of the monograph [7], one could easily treat this more general case. One can see, e.g., that $\langle e^{t\xi}\rangle = e^{\frac{t^{\alpha'}}{\alpha'}\ell(t)}$, where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and $\ell(t)$ is again a normal slowly varying function, related to $L$ by a Legendre transform, up to equivalence.

For applications of the results of [7] to limit theorems, in a setting similar to ours, see [8].

In this paper, in order to keep technicalities to a minimum, we assume that the random variables $\{V(x) : x \in \mathbb{Z}^d\}$ are i.i.d. and such that the quantities $V^2(x)/2$ have a Weibull distribution (1.4).

Our central object is the (formal) parabolic equation

$$\frac{\partial u}{\partial t} = Hu := \kappa\Delta u + V(x,\omega)\dot{w}_t u, \qquad u(0,x) \equiv 1, \quad t \geq 0,\ x \in \mathbb{Z}^d. \tag{1.5}$$

We give a precise mathematical meaning to the stochastic partial differential equation (1.5) in some appropriate functional space, by stating its equivalence to the integral equation

$$u(t,x) = 1 + \kappa \int_0^t \Delta u(s,x)ds + V(x,\omega)\int_0^t u(s,x) \circ dw_s, \tag{1.6}$$

where the last integral is understood in the Stratonovich sense. Such interpretation, as shown in Sect. 3, leads to the Feynman–Kac representation of the solution

$$u(t,x) = \mathbb{E}_x e^{\int_0^t V(X_s,\omega)dw_{t-s}}, \tag{1.7}$$

where $X_t, t \geq 0$ is the continuous-time random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$, and $\mathbb{E}_x$ denotes expectation with respect to the random walk $X_s$ starting at $x$: $X_0 = x$.

Questions of existence and uniqueness are deferred to Sect. 3.

*Remark* As $X_s$ is a random walk on $\mathbb{Z}^d$, $V(X_s,\omega)$ is piecewise constant and takes a finite number of values, a.e. with respect to the random walk measure. So there is no difference between the Ito and Stratonovich interpretations of the integral $\int_0^t V(X_s,\omega)dw_{t-s}$.

The fast decay of the tails of the potential distribution (1.4) ensures, as we will see, that all moments of the solution are finite for all $t$. We will consider the annealed moments

$$m_p(t) = \langle u^p(t,x)\rangle = \langle\langle u^p(t,x)\rangle_{P_V}\rangle_{P_w}, \quad p = 1,2,\ldots \tag{1.8}$$

which clearly do not depend on $x$, and the semi-annealed ones, corresponding to taking expectation only with respect to the white noise. The latter moments, by Gaussian integration, are expressed as

$$\begin{aligned} U_p(t,x_1,x_2,\ldots,x_p) &:= \langle u(t,x_1)u(t,x_2)\ldots u(t,x_p)\rangle_{P_w} \\ &= \mathbb{E}_{x_1,\ldots,x_p} e^{\frac{1}{2}\int_0^t [V(X_s^{(1)})+\cdots+V(X_s^{(p)})]^2 ds}, \end{aligned} \tag{1.9}$$

where $X_s^{(1)},\ldots,X_s^{(p)}$ are i.i.d. copies of the random walk, and $\mathbb{E}_{x_1,\ldots,x_p}$ is the expectation for fixed initial positions. Clearly $m_p(t) = \langle U_p(t,0,\ldots,0)\rangle_{P_V}$.

The aim of this paper is to find the asymptotic behavior, as $t \to \infty$, of the annealed and semi-annealed moments $m_p(t)$ and $U_p(t,0,\ldots,0)$ of the solution $u(t,\cdot)$. Our main results are the following.

**Theorem 1** *Under the assumptions above, if* $H(t) := \ln\langle e^{t\frac{V^2(0)}{2}}\rangle$, *then, as* $t \to \infty$, *the following asymptotics holds*

$$m_p(t) = e^{H(\frac{p^2 t}{2}) - p\kappa t + o(t)}, \quad p = 1,2,\ldots. \tag{1.10}$$

As for the semi-annealed moments we have an almost-sure result.

**Theorem 2** *Under the assumptions above, if $C_d = (\frac{\alpha d}{c})^{\frac{1}{\alpha}}$, the following asymptotics holds, as $t \to \infty$, for $P_V$-almost all realizations of the potential,*

$$U_p(t, 0, 0, \dots, 0) = e^{C_d p^2 t \ln^{\frac{1}{\alpha}} t - p\kappa t + o(t)}. \tag{1.11}$$

In the proof of both Theorems 1 and 2 we will make use of the fact that, as it should be expected for Weibull variables on the basis of the discussion in [2], the main contribution to the moments comes from single peaks of $|V(x, \omega)|$.

Proofs are given in Sects. 3 and 4. Section 2 contains some auxiliary results. The fully quenched case is briefly treated in Sect. 5 in a simplified finite-dimensional version, which is however a good illustration of the behavior of the real one.

The analysis of the fully quenched behavior of the original model is technically rather complex and will be the object of future work.

Throughout the paper $|\cdot|$ will denote the euclidean norm for points of $\mathbb{R}^d$ or $\mathbb{Z}^d$.

## 2 Some Technical Lemmas

**Lemma 2.1** *If $\xi$ is the Weibull variable (1.4), then, as $t \to \infty$, we have*

$$H_{(\alpha,c)}(t) = \ln\langle e^{t\xi}\rangle = \frac{tv_*(t)}{\alpha'} + \frac{\alpha}{2}\ln v_*(t) + \frac{1}{2}\ln\frac{2\pi c}{\alpha - 1} + o(1), \tag{2.1}$$

*where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ and $v_*(t) = (\frac{t}{c})^{\frac{1}{\alpha-1}}$. Moreover for any integer $m \geq 0$,*

$$\langle \xi^m e^{t\xi}\rangle = v_*^m(t)e^{H_{(\alpha,c)}(t)}(1 + o(1)). \tag{2.2}$$

*Proof* Integrating by parts and introducing the function $g_\alpha(x) = x - \frac{x^\alpha}{\alpha}$, we find

$$\begin{aligned}\langle \xi^m e^{t\xi}\rangle &= -\int_0^\infty x^m e^{tx}\frac{d}{dx}e^{-c\frac{x^\alpha}{\alpha}}dx = \delta_{m0} + \int_0^\infty x^{m-1}(m + tx)e^{tx - c\frac{x^\alpha}{\alpha}}dx \\ &= \delta_{m0} + v_*^m(t)\int_0^\infty e^{\lambda g_\alpha(y)}(m + \lambda y)y^{m-1}dy,\end{aligned}$$

where we have set $x = v_*(t)y$ and $\lambda = tv_*(t) = c(\frac{t}{c})^{\alpha'}$. As $\lambda \to \infty$, we apply the Laplace method. The function $g_\alpha(x)$ has a proper maximum at $x = 1$, with $g_\alpha(1) = \frac{1}{\alpha'}$, and $g_\alpha''(1) = -\alpha + 1$. Hence, by a standard result,

$$\langle \xi^m e^{t\xi}\rangle = v_*^m(t)\sqrt{\frac{2\pi\lambda}{\alpha - 1}}e^{\frac{tv_*(t)}{\alpha'}}(1 + o(1)). \tag{2.3}$$

For $m = 0$ we get (2.1), and for $m > 0$, expressing the right side of (2.3) in terms of $e^{H_{(\alpha,c)}(t)}$, we get (2.2). □

The proof of Lemma 2.1 shows that the main contribution comes from around the point $x = v_*(t)$. This fact is made more precise by the following lemma.

**Lemma 2.2** *Under the assumptions above we have, as $t \to \infty$, for any $\delta > 0$,*

$$e^{H_{(\alpha,c)}(t)} \sim \langle e^{t\xi} \mathbb{I}_{|\xi - v_*(t)| < \delta v_*(t)} \rangle.$$

*Proof* In fact, by the previous arguments we see that

$$e^{-\frac{\lambda}{\alpha'}} \langle e^{t\xi} \mathbb{I}_{|\xi - v_*| \geq \delta v_*} \rangle \leq e^{-\frac{\lambda}{\alpha'}} + \lambda \int_{|x-1| \geq \delta} e^{-\lambda(g_\alpha(1) - g_\alpha(x))} dx \leq c_\delta e^{-\lambda s_\delta},$$

where $c_\delta$ and $s_\delta$ are positive constants with $s_\delta \in (0, g_\alpha(1) - g_\alpha(1-\delta))$, and we made use of the fact that $g_\alpha(1) - g_\alpha(x)$ is a positive convex function with minimum at $x = 1$. □

**Lemma 2.3** *Let $\{\tilde{V}(x) : x \in \mathbb{Z}^d\}$ be i.i.d. variables with common Weibull distribution (1.4). Then, for any choice of $a > d^{-1}$ one can find almost-surely a positive number $R_0$, depending on the realization, such that the inequalities*

$$C_d(\ln R - a \ln\ln R)^{\frac{1}{\alpha}} \leq \max_{|x| \leq R} U(x) \leq C_d(\ln R + a \ln\ln R)^{\frac{1}{\alpha}}, \tag{2.4}$$

*hold, for $R > R_0$, with $C_d = (\frac{\alpha d}{c})^{\frac{1}{\alpha}}$.*

*Proof* By (1.4), setting $\ln_+(\cdot) = \max\{\ln(\cdot), 1\}$, we find

$$P(\{\tilde{V}(x) > C_d(\ln_+ |x| + a \ln_+ \ln_+ |x|)^{\frac{1}{\alpha}}\}) = \exp\{-d \ln_+ |x| - ad \ln_+ \ln_+ |x|\}.$$

The right side is summable, as $ad > 1$, and, by the first Borel–Cantelli lemma, we have a.s., as $|x| \to \infty$, $\tilde{V}(x) \leq C(d \ln |x| + a \ln\ln |x|)$, which implies the right inequality (2.4). Let now $n_R$ be the cardinality of the set $\{x \in \mathbb{Z}^d : |x| \leq R\}$. Then for $R$ large

$$P\left(\left\{\max_{|x| \leq R} \tilde{V}(x) < C_d(\ln R - a \ln\ln R)^{\frac{1}{\alpha}}\right\}\right) = \left(1 - \frac{\ln^{ad} R}{R^d}\right)^{n_R} \sim \exp\{-B_d \ln^{ad} R\}, \tag{2.5}$$

where $B_d = \lim_{R\to\infty} \frac{n_R}{R^d}$ is the volume of the unit sphere in $\mathbb{R}^d$. As $ad > 1$ the series $\sum_{x\in\mathbb{Z}^d} e^{-B_d \ln_+^{ad} |x|}$ converges, so that, again by the Borel–Cantelli lemma, the left inequality (2.4) is eventually satisfied. □

## 3 Existence and Uniqueness Theorem

In this paragraph we will give existence and uniqueness results for the parabolic Anderson model (1.5) with quasi-stationary potential, i.e., for the integral equation (1.6). $V(x)$ will always denote a single (deterministic) function. We introduce a weighted Hilbert space $L^2_\sigma$, for $0 < \sigma < 1$, of functions on $\mathbb{Z}^d$, and, for any $T > 0$ a corresponding Hilbert space $H_{T,\sigma}$ of functions $u : [0, T] \times \mathbb{Z}^d \to L^2_\sigma$, depending on the Brownian motion $w$. The corresponding inner products are

$$\begin{aligned} (f, g)_\sigma &= \sum_{x\in\mathbb{Z}^d} f(x)\bar{g}(x)\sigma^{|x|}, \\ (u, v)_{T,\sigma} &= \int_0^T \langle (u(s, \cdot), v(s, \cdot))_\sigma \rangle_{P_w} ds, \end{aligned} \tag{3.1}$$

while $\|\cdot\|_\sigma$ and $\|\cdot\|_{T,\sigma}$ denote the corresponding norms.

**Theorem 3.1** *If $V$ is a bounded function on $\mathbb{Z}^d$, then* (1.6) *has, for all finite $T$, $P_w$-a.e., a unique solution $u \in H_{T,\sigma}$, such that $P_w$-a.e. $u(t,x) \in L^2_\sigma$ is continuous in* $[0,T]$. *Moreover $u(t,x)$ can be represented in the form* (1.7).

*Proof* It is well known that the Stratonovich integral in (1.6) can be written as

$$\int_0^t u(s,x) \circ dw_s = \int_0^t u(s,x)dw_s + \frac{V(x)}{2}\int_0^t u(s,x)ds,$$

where the first integral on the right is an Ito integral. Equation (1.6) then becomes

$$u(t,x) = 1 + \int_0^t Au(s,x)ds + V(x)\int_0^t u(s,x)dw_s, \tag{3.2}$$

where $Au(t,x) = \kappa\Delta u(t,x) + \frac{V^2(x)}{2}u(t,x)$. $A$ is a bounded linear operator on $L^2_\sigma$, with norm $\|A\|$ depending on $\kappa, \sigma$, and $M =: \sup_x |V(x)|$.

To (3.2) we now apply the usual iteration scheme to prove existence: we set

$$u_{n+1}(t,x) = 1 + \int_0^t Au_n(s,x)ds + V(x)\int_0^t u_n(s,x)dw_s, \tag{3.3}$$

with initial point $u_0 \equiv 1$. Passing to the differences $v_n(t,x) = u_n(t,x) - u_{n-1}(t,x)$, with $v_0 = u_0$, by standard arguments (see [10]), one finds for any $T > 0$ and all $t \in [0,T]$, the inequalities

$$\langle \|v_{n+1}(t,\cdot)\|^2_\sigma\rangle_{P_w} \le K_T \int_0^t \langle \|v_n(s,\cdot)\|^2_\sigma\rangle_{P_w} ds, \quad K_T = 2(T\|A\|^2 + M^2), \tag{3.4}$$

implying that $\|v_n(t,\cdot)\|^2_\sigma \le C_0 \frac{(tK_T)^n}{n!}$. Integrating in $t$ we get convergence of the series $u(t,x) = 1 + \sum_{n=1}^\infty v_n(t,x)$ in the Hilbert space $H_{T,\sigma}$.

A.s. convergence is given by a Borel–Cantelli argument (see [4]). We have

$$\begin{aligned} &P\Big(\max_{0\le t\le T} \|v_n(t,\cdot)\|^2_\sigma > 2^{-n}\Big) \\ &\quad \le P\bigg(2T\int_0^T \|Av_{n-1}(s,\cdot)\|^2_\sigma ds > \frac{1}{2^{n+1}}\bigg) \\ &\qquad + P\bigg(2M^2 \sum_x \sigma^{|x|} \max_{0\le t\le T}\bigg(\int_0^t v_{n-1}(s,x)dw_s\bigg)^2 > \frac{1}{2^{n+1}}\bigg). \end{aligned}$$

Applying to the second term the well-known inequality $\langle \max_{0\le t\le T}(\int_0^t v(s)dw_s)^2\rangle_{P_w} \le 4\int_0^T \langle v^2(s)\rangle_{P_w} ds$ it is easy to see that for some constant $\bar{C}$

$$P_w\Big(\max_{0\le t\le T} \|v_n(t,\cdot)\|^2_\sigma > 2^{-n}\Big) \le \bar{C}\frac{(TK_T)^n}{n!}.$$

As the right side is summable, again by Borel–Cantelli, we see that for some non-negative random variable $\xi$, $\max_{0\le t\le T} \|v_n(t,\cdot)\|_\sigma \le \xi 2^{-\frac{n}{2}}$, $P_w$-a.e. Therefore $u(t,x)$, as a limit of continuous functions on $[0,T]$ with values in $L^2_\sigma$ which converge uniformly, is continuous.

It is now easy to check that $u(t,x)$ satisfies (3.2) in $L^2_\sigma$.

As for uniqueness, if $v=u_1-u_2$ is the difference of two solutions, repeating the steps that led to inequality (3.4) we get a Gromwall inequality for $v$, and hence, by a standard procedure, uniqueness. We omit the details.

The proof that the solution is expressed by the Feynman–Kac formula (1.7) can be done, as in [4] by taking a regularization of the white noise, i.e.,

$$\dot{w}_\delta(t)=\frac{w_\delta}{\delta}I_{[0,\delta)}(t)+\frac{w_{2\delta}-w_\delta}{\delta}I_{[\delta,2\delta)}(t)+\cdots \tag{3.5}$$

which leads to the regularized equation

$$\frac{\partial u_\delta(t,x)}{\partial t}=\kappa\Delta u_\delta(t,x)+\dot{w}_\delta(t)V(x)u_\delta(t,x),\qquad u_\delta(0,x)=1. \tag{3.6}$$

The Feynman–Kac solution $u_\delta(t,x)=\mathbb{E}_x e^{\int_0^t \dot{w}_\delta(s)V(X_{t-s})ds}$ is the unique solution of the problem (3.6) in $L^2_\sigma$ (see [4]). Let $u_\delta(t,x)=\mathbb{E}_x F_\delta(t)$, $u(t,x)=\mathbb{E}_x F(t)$, with

$$F_\delta(t)=e^{\int_0^t \dot{w}_\delta(t-s)V(X_s)ds},\qquad F(t)=e^{\int_0^t V(X_s)dw_{t-s}}.$$

If $\mathcal{P}_x$ denotes the probability associated to the random walk starting at $x$, the function $V(X_s)$, for $s\in[0,t]$ takes finitely many values $\mathcal{P}_x$-a.e., so that, as $\delta\to 0$, $F_\delta(t)\to F(t)$, $\mathcal{P}_x\times P_w$-a.e. By Gaussian integration we have, for $p\ge 1$,

$$\langle|u(t,x)|^p\rangle_{P_w}\le \mathbb{E}_x\langle(F(t))^p\rangle_{P_w}=\mathbb{E}_x e^{\frac{p^2}{2}\int_0^t V^2(X_s)ds}, \tag{3.7}$$

$$\langle|u_\delta(t,x)|^p\rangle_{P_w}\le \mathbb{E}_x\langle(F_\delta(t))^p\rangle_{P_w}=\mathbb{E}_x e^{\frac{p^2}{2\delta}\sum_{k=0}^{n-1}(\int_{k\delta}^{(k+1)\delta}V(X_s)ds)^2} \tag{3.8}$$

(we take for simplicity $\delta=t/n$, for some integer $n>1$). By pointwise convergence and the uniform bound (3.8), valid for all $p\ge 1$, it is easy to see (using, e.g., the Egorov theorem), that, as $\delta\to 0$, $F_\delta(t)\to F(t)$ in $L_p(\mathcal{P}_x\times P_w)$, and hence $u_\delta(t,x)\to u(t,x)$ in $L_p(P_w)$. For $p=2$ convergence in $L_2(P_w)$ and the uniform bound in $x$ give

$$\lim_{\delta\to 0}\max_{t\in[0,T]}\langle\|u(t,\cdot)-u_\delta(t,\cdot)\|_\sigma^2\rangle_{P_w}=0, \tag{3.9}$$

which implies convergence of $u_\delta$ in $H_{T,\sigma}$.

As for the equation that is satisfied by the limiting function $u$, observe that

$$\begin{aligned}\int_0^t \dot{w}_\delta(s)u_\delta(s,x)ds&=\sum_{k=0}^{n-1}(w_{(k+1)\delta}-w_{k\delta})\\&\quad\times\left[u_\delta(k\delta,x)+\frac{1}{\delta}\int_{k\delta}^{(k+1)\delta}[u_\delta(s,x)-u_\delta(k\delta,x)]ds\right].\end{aligned}$$

The difference in the integral on the right is expressed, by (3.6), in integral form:

$$u_\delta(t+h,x)-u_\delta(t,x)=\kappa\int_t^{t+h}\Delta u_\delta(s,x)ds+V(x)\int_t^{t+h}\dot{w}_\delta(s)u_\delta(s,x)ds.$$

Substituting and taking the limit as $\delta\to 0$, one recovers all terms of (3.2).

Theorem 3.1 is proved. □

If $V$ is unbounded, we introduce, for any $M>0$, the truncated potential $V_M(x)=\frac{V(x)}{|V(x)|}\min\{|V(x)|,M\}$ and the random elements

$$F^{(M)}(t)=e^{\int_0^t V_M(X_s)dw_{t-s}},\qquad u^{(M)}(t,x)=\mathbb{E}_x e^{\int_0^t V_M(X_s)dw_{t-s}}.$$

$u(t,x)$ will denote the expression (1.7). By the previous theorem, $u^{(M)}(t,x)\in L^2_\sigma$ is the unique solution of (3.2) with potential $V_M$.

**Theorem 3.2** *Suppose that $V^2(x)\le C(1+|x|^\delta)$, for some constants $C>0$ and $\delta\in(0,1)$. Then, under the assumptions above, the following assertions hold.*

- (i) *For all $p\ge 1$, $u(t,x)\in L_p(P_w)$, and, as $M\to\infty$, for all $(t,x)\in\mathbb{R}_+\times\mathbb{Z}^d$, the functions $u^{(M)}(t,x)$ converge to $u(t,x)$, in $L_p(P_w)$, as well as $P_w$-a.e.*
- (ii) *For any $T>0$, $u(t,x)\in H_{T,\sigma}$ and satisfies (3.2) as an equation in $L^2_\sigma$ $P_w$-a.e.*
- (iii) *The solution of (3.2) is unique in the class of the functions of $H_{T,\sigma}$ which are such that, for any integer $p\ge 1$, $e^{-\bar c_p|x|^\alpha}\max_{t\in[0,T]}\langle(u(t,x))^p\rangle_{P_w}$ is bounded in $x$, for some $\alpha\in(0,1)$ and some positive constants $\bar c_p$.*

*Proof* As before, observe first, that, $\mathcal{P}_x\times P_w-a.e.$, $F^{(M)}(t)\to F(t)$. Proceeding as for (3.7) we see that for all $p\ge 1$ we have

$$\langle(u^{(M)}(t,x))^p\rangle_{P_w}\le\mathbb{E}_x e^{\frac{p^2}{2}\int_0^t V_M^2(X_s)ds}\le\mathbb{E}_x e^{\frac{p^2}{2}\int_0^t V^2(X_s)ds}=\mathbb{E}_x\langle(F(t))^p\rangle_{P_w}.\tag{3.10}$$

Let $n_t$ be the number of jumps of the random walk up to time $t$ (which is Poisson distributed with mean $\kappa t$). Then $\sup_{0\le s\le t}|X_s|\le|x|+n_t$ and $\sup_{0\le s\le t}V^2(X_s)\le C(1+(|x|+n_t)^\delta)\le C(1+|x|^\delta+n_t^\delta)$, so that

$$\mathbb{E}_x e^{\frac{p^2}{2}\int_0^t V^2(X_s)ds}\le e^{\frac{Cp^2t}{2}(1+|x|^\delta)}\mathbb{E}(e^{\frac{Cp^2t}{2}n_t^\delta})=C_{p,t}e^{\frac{Cp^2t}{2}|x|^\delta},\tag{3.11}$$

where $\mathbb{E}$ denotes the average over $n_t$, which is finite as $\delta\in(0,1)$. Therefore the last term in (3.10) is bounded and $u^{(M)}(t,x)\to u(t,x)$ in $L_p(P_w)$ for all $p\ge 1$. For convergence $P_w$-a.e., let $n_M$ be the smallest integer $n\ge 0$ such that $C(1+(|x|+n)^\delta)\ge M^2$. If $n_T<n_M$, as $X_0=x$, $V(X_t)=V_M(X_t)$ for $t\in[0,T]$, setting $\hat u^{(M)}(t,x)=u(t,x)-u^{(M)}(t,x)$, we have

$$\begin{aligned}&\|\hat u^{(M)}(t,x)\|^2_{L_2(P_w)}\\&\quad\le 2\mathbb{E}_x(e^{2\int_0^t V^2(X_s)ds};n_T\ge n_M)\le 2e^{2CT(1+|x|^\delta)}\mathbb{E}(e^{2CTn_T^\delta};n_T\ge n_M)\\&\quad\le 2C_{2,T}e^{2CT|x|^\delta}N_T(n_M),\end{aligned}\tag{3.12}$$

where $N_T(n)=e^{2CTn^\delta}(\kappa T)^n/n!$, and the last inequality comes from the easy estimate $\mathbb{E}(e^{2CTn_T^\delta};n_T\ge n)\le\frac{e^{2CTn^\delta}(\kappa T)^n}{n!}\mathbb{E}(e^{2CTn_T^\delta})$. As $N_T(n)$ is summable, we get convergence a.e., and assertion (i) is proved.

Let $R=(\frac{M^2}{C}-1)^{\frac{1}{\delta}}$, so that $n_M+|x|\ge R$, and set $n_0=[\frac{R}{2}]$, where $[\cdot]$ denotes the integer part. If $|x|<n_0$, then $n_M\ge n_0$, so that, for $M$ (or $n_0$) large we have, by (3.12),

$$\begin{aligned}\langle\|\hat u^{(M)}(t,\cdot)\|^2_\sigma\rangle_{P_w}&\le C_T^{(1)}N_T(n_0)\sum_{|x|<n_0}e^{2CT|x|^\delta}\sigma^{|x|}+C_T^{(2)}\sum_{|x|\ge n_0}e^{2CT|x|^\delta}\sigma^{|x|}\\&\le\bar C_T\max\{N_T(n_0),\sigma^{\frac{n_0}{2}}\}.\end{aligned}\tag{3.13}$$

As the right side is summable, we have convergence $P_w$-a.e. in $L^2_\sigma$, and $u \in H_{T,\sigma}$.

To show that (3.2) is satisfied, consider the difference of the right sides for $u$ and for $u^{(M)}$. It is a sum of three terms: $D^{(1)}_M(t,x) + D^{(2)}_M(t,x) + D^{(3)}_M(t,x)$:

$$D^{(1)}_M(t,x) = \int_0^t \Delta \hat{u}^{(M)}(s,x)ds,$$

$$D^{(2)}_M(t,x) = \frac{V_M^2(x)}{2}\int_0^t u^{(M)}(s,x)ds - \frac{V^2(x)}{2}\int_0^t u(s,x)ds,$$

$$D^{(3)}_M(t,x) = V(x)\int_0^t u(s,x)dw_s - V_M(x)\int_0^t u^{(M)}(s,x)dw_s.$$

For the expected value of the $L^2_\sigma$-norm of the first term we find, by (3.13)

$$\begin{aligned}\left\langle \sup_{t\in[0,T]} \|D^{(1)}_M(t,\cdot)\|^2_\sigma \right\rangle_{P_w} &\le T\int_0^T \langle \|\Delta \hat{u}^{(M)}(t,\cdot)\|^2_\sigma\rangle_{P_w} ds \\ &\le C_T\|\Delta\|^2 \max\{N_T(n_0), \sigma^{\frac{n_0}{2}}\},\end{aligned}$$

where $\|\Delta\|$ is the operator norm of the Laplacian (1.2) in $L^2_\sigma$. By the usual Borel–Cantelli argument this term vanishes $P_w$-a.e. as $M \to \infty$.

Similar considerations give the same result for the other two terms as well. We leave the details to the reader. Assertion (ii) is proved.

To prove uniqueness, we use a truncation procedure. Let $B_R = \{x \in \mathbb{Z}^d : |x| \le R\}$, and let $\partial B_R = \{x \in \mathbb{Z}^d \setminus B_R : \min_{y\in B_R}|x-y| = 1\}$ denote its boundary. We consider the solution of (3.2) with initial values $u_0(x), x \in B_R$ and boundary conditions $u_0(t,x), x \in \partial B_R$, and clearly any solution of (3.2) can be represented as the solution of the truncated problem, by assigning the appropriate boundary values. The difference of any two solutions $v(t,x) = u_2(t,x) - u_1(t,x)$ with the same initial conditions is represented in the form

$$v(t,x) = \mathbb{E}_x[e^{\int_0^{\tau_R} V(X_s)dw_{t-s}} v(t-\tau_R, X_{\tau_R})\mathbb{I}_{\tau_R \le t}],$$

where $\tau_R$ is the first hitting time of the set $\partial B_R$ (this is a consequence of the strong Markov property, see [4]). Observe that the potential $V$ is bounded in $B_R$, so we have a unique solution as in Theorem 3.1 [4].

By the Schwartz and Hölder inequalities, recalling the bound on the moments of $v(t,x)$, repeating the steps that led to inequality (3.12), we get

$$\begin{aligned}\langle v^2(t,x)\rangle_{P_w} &\le \mathbb{E}_x(\langle e^{2\int_0^{\tau_R} V(X_s)dw_{t-s}} v^2(t-\tau_R, X_{\tau_R})\rangle_{P_w}; \tau_R \le t) \\ &\le \max_{s\in[0,t]}\max_{y\in\partial B_R} \langle v^4(s,y)\rangle^{\frac{1}{2}}_{P_w} \mathbb{E}_x(e^{8\int_0^{\tau_R} V^2(X_s)ds}; n_t \ge R) \\ &\le c_0 \exp\{c_1 R^\alpha + c_2 R^\delta - c_3 R\},\end{aligned}$$

for some positive constants $c_0, \dots, c_3$ depending on $t$. As $R$ is arbitrary, it must be $\langle \|v(t,\cdot)\|^2_\sigma\rangle_{P_w} = 0$. Theorem 3.2 is proved. □

**Corollary 3.3** *All moments of the solution are finite and can be written in the form*

$$U_p(t,x_1,\dots,x_p) = \langle u(t,x_1)\cdot \dots \cdot u(t,x_p)\rangle_{P_w} = \mathbb{E}_{x_1\cdots x_p} e^{\frac{1}{2}\int_0^t (V(X_s^1)+\dots+V(X_s^p))^2 ds} \qquad (3.14)$$

*where $\mathbb{E}_{x_1\cdots x_p}$ denotes expectation with respect to $p$ independent random walks on $\mathbb{Z}^d$ with initial positions $x_1, \dots, x_p$. The moments* (3.14) *satisfy the initial value problem*

$$\frac{\partial U_p}{\partial t} = \kappa(\Delta_{x_1} + \cdots + \Delta_{x_p})U_p + \frac{1}{2}\left(\sum_{i=1}^{p} V(x_i)\right)^2 U_p,$$
$$U_p(0, x_1, \dots, x_p) = 1. \tag{3.15}$$

*Proof* The first assertion follows from the representation (1.7) and the usual rules of integration over the Brownian motion. Equation (3.15) is a consequence of (3.14). □

*Remark* Theorem 3.2 depends in an essential way on the choice of the initial condition $u(0,x) \equiv 1$. The initial condition should not be fast increasing in $x$. For a general $u(0,x) \in L^2_\sigma(\mathbb{Z}^d)$ the solution of (3.2) with unbounded $V(x)$ does not exist.

## 4 Proofs of Theorems 1 and 2

Throughout this paragraph the random variables $\{V(x) : x \in \mathbb{Z}^d\}$ are i.i.d. and such that $V^2(x)/2$ has the Weibull distribution (1.4) for some $\alpha > 1$. By Lemma 2.3, and Theorem 3.2 for a.a. realizations of $\{V(x) : x \in \mathbb{Z}^d\}$ there is a unique solution $u(t,x)$ of (3.2), given by the Feynman–Kac formula (1.7).

The following preliminary lemma, is analogous to the corresponding result in [1].

**Lemma 4.1** *For all $p \geq 1$ and $t \geq 0$ the following inequalities hold*

$$e^{H(p^2 t) - \kappa p t} \leq m_p(t) \leq e^{H(p^2 t)}. \tag{4.1}$$

*Proof* By the usual Gaussian integration formula we have

$$\begin{aligned}\langle (u(t,0))^p \rangle_{P_w} &= \langle \mathbb{E}_{0,\dots,0}[e^{\int_0^t \sum_{j=1}^p V(X_s^{(j)}) dw_{t-s}}] \rangle_{P_w} \\ &= E_{0,\dots,0}[e^{\frac{1}{2}\int_0^t (\sum_{j=1}^p V(X_s^{(j)}))^2 ds}].\end{aligned} \tag{4.2}$$

If now $n_t^{(j)}$, $j = 1, \dots, p$ are the number of jumps, up to time $t$, of the $p$ independent random walks, with initial condition $X_0^{(j)} = 0$, $j = 1, \dots, p$, the simple inequality

$$\begin{aligned}m_p(t) = \langle (u(t,0))^p \rangle &= \langle \mathbb{E}_{0,\dots,0}[e^{\frac{1}{2}\int_0^t (\sum_{j=1}^p V(X_s^{(j)}))^2 ds}] \rangle_{P_V} \\ &\geq \langle e^{\frac{p^2 t}{2} V^2(0)} \rangle_{P_V} \prod_{j=1}^{p} P(n_t^{(j)} = 0) = e^{H(p^2 t) - \kappa p t},\end{aligned} \tag{4.3}$$

gives the bound on the left of (4.1). Moreover, by the Jensen inequality, we have

$$\begin{aligned}e^{\frac{1}{2}\int_0^t (\sum_{j=1}^p V(X_s^{(j)}))^2 ds} &= e^{\frac{1}{2t}\int_0^t t(\sum_{j=1}^p V(X_s^{(j)}))^2 ds} \\ &\leq \frac{1}{t}\int_0^t e^{\frac{t}{2}(\sum_{j=1}^p V(X_s^{(j)}))^2} ds \leq \frac{1}{t}\int_0^t e^{\frac{pt}{2}\sum_{j=1}^p V^2(X_s^{(j)})} ds,\end{aligned}$$

and it is now easy to get the bound on the right

$$
\begin{aligned}
&\langle E_{0,\dots,0}[e^{\frac{1}{2}\int_0^t(\sum_{j=1}^p V(X_s^{(j)}))^2 ds}]\rangle_{P_V} \\
&\quad \le \frac{1}{t} E_{0,\dots,0}\int_0^t \langle e^{\frac{pt}{2}\sum_{j=1}^p V^2(X_s^{(j)})}\rangle_{P_V} ds \\
&\quad \le \frac{1}{t} E_{0,\dots,0}\int_0^t \left(\prod_{j=1}^p \langle e^{\frac{p^2 t}{2} V^2(X_s^{(j)})}\rangle_{P_V}\right)^{\frac{1}{p}} ds = e^{H(p^2 t)}.
\end{aligned}
$$

Lemma 4.1 is proved. □

*Remark* Lemma 4.1 already implies intermittency for the annealed model in all dimension $d \ge 1$ (see [4]).

In the second step, as in Theorem 3.2, we control the growth of the potential by a bound on the number of jumps. We set $\bar{n}_t = \max\{n_t^{(j)}, j = 1, \dots, p\}$, where $n_t^{(j)}$, $j = 1, \dots, p$ are as in the proof of Lemma 4.1. The maximum of $V^2(x)/2$ on $B_R = \{|x| \le R\}$ is denoted $M_R := \max\{V^2(x)/2 : |x| \le R\}$.

**Lemma 4.2** *Under the assumptions above, if $r_t = [b\kappa t]$, for $b(\ln b - 1) > p$, where $[\cdot]$ denotes the integer part, we have, as $t \to \infty$,*

$$
m_p(t) \sim \langle \mathbb{E}_{0,\dots,0}[\mathbb{I}_{\{\bar{n}_t \le r_t\}} e^{\frac{1}{2}\int_0^t(\sum_{j=1}^p V(X_s^{(j)}))^2 ds}]\rangle_{P_V}. \tag{4.4}
$$

*Proof* Clearly $P(\bar{n}_t = r) \le \sum_{j=1}^p P(n_t^{(j)} = r) = pP(n_t = r)$, where $n_t$ is Poisson distributed with mean $\kappa t$. The contribution of large values of $\bar{n}_t$ is then bounded by

$$
\begin{aligned}
&\sum_{r=r_t+1}^{\infty} \langle \mathbb{E}_{0,\dots,0}[\mathbb{I}_{\{\bar{n}_t = r\}} e^{\frac{1}{2}\int_0^t(\sum_{k=1}^p V(X_s^{(k)}))^2 ds}]\rangle_{P_V} \\
&\quad \le \sum_{r=r_t+1}^{\infty} P(\bar{n}_t = r)\langle e^{tp^2 M_r}\rangle_{P_V} \le p \sum_{r=r_t+1}^{\infty} P(n_t = r)\langle e^{tp^2 M_r}\rangle_{P_V}.
\end{aligned} \tag{4.5}
$$

Simple geometric considerations give, for some constant $K_d$,

$$
\langle e^{p^2 M_r t}\rangle_{P_V} = \sum_{|x| \le r} \langle e^{tp^2 \frac{V^2(x)}{2}} \mathbb{I}_{\{M_r = \frac{V^2(x)}{2}\}}\rangle_{P_V} \le K_d r^d e^{H(p^2 t)}. \tag{4.6}
$$

The right side of (4.5) is then bounded by $pK_d e^{H(p^2 t)}\mathbb{E}(n_t^d; n_t > r_t)$. We have $\mathbb{E}(n_t^d; n_t > r_t) \le \frac{(\kappa t)^{r_t}}{r_t!}\mathbb{E}[(r_t + n_t)^d]$ and, for large $t$, $\mathbb{E}[(r_t + n_t)^d] \le \tilde{c}_d(\kappa t + r_t)^d$. Therefore the right side of (4.5) is bounded by the expression

$$
\begin{aligned}
&\exp\{H(p^2 t) + r_t \ln \kappa t - r_t \ln r_t + r_t + \mathcal{O}(\ln \kappa t)\} \\
&\quad = \exp\{H(p^2 t) - \kappa t b(\ln b - 1) + \mathcal{O}(\ln t)\},
\end{aligned}
$$

which, by our assumption on $b$, is $o(e^{H(p^2 t) - \kappa p t})$.

By the result of Lemma 4.1, the proof of Lemma 4.2 is achieved. □

We now turn to the proof of Theorem 1. We make use of the fact that, by Lemma 2.2, the main contribution to the integral $\langle e^{t\frac{V^2(x)}{2}}\rangle$ comes from a small interval around the point $\frac{V^2(x)}{2} = v_*(t)$.

*Proof of Theorem 1* We will prove that $\ln m_p(t) \le H(p^2 t) - \kappa p t + o(t)$, which, taking into account Lemma 4.1, completes the proof of Theorem 1. As in [5], the proof is based on the spectral analysis of the operator on the right side of (3.15).

We consider only the case $p = 1$, as there is no difficulty in extending the result to any $p > 1$. We set for brevity $U(t) := U_1(t, 0)$, where $U_1(t, x)$ is the solution of (3.15) for $p = 1$, and is given by the Feynman–Kac formula (1.9):

$$U(t) = \mathbb{E}_0 e^{\frac{1}{2}\int_0^t V^2(X_s)ds}, \tag{4.7}$$

which is basic for our proofs. Consider the discrete cube $S_t^d = \{x \in \mathbb{Z}^d : |x_i| \le r_t, i = 1, \dots, d\}$, where $r_t$ is as in the proof of Lemma 4.2. Proceeding as in [5], we denote by $\tilde{U}(t, x)$ the solution of (3.15) for $p = 1$, and with periodic boundary conditions on $S_t^d$. $\tilde{U}(t, x)$ is a solution of the problem

$$\frac{\partial \tilde{U}(t, x)}{\partial t} = \kappa \Delta \tilde{U}(t, x) + \frac{V^2(x)}{2}\tilde{U}(t, x), \qquad \tilde{U}(t, 0) \equiv 1, \quad x \in S_t^d \tag{4.8}$$

on the "discrete torus $S_t^d$", obtained by identifying the pairs of opposite sides, and is given by the same Feynman–Kac formula (4.7) except that the potential $V$ is replaced by a new potential $\hat{V}$ which is defined by setting $\hat{V}(x) = V(x)$ for $x \in S_t^d$, and is extended by periodicity to all $\mathbb{Z}^d$. Consider now the truncated solutions

$$U_*(t) = \mathbb{E}_0(e^{\frac{1}{2}\int_0^t V^2(X_s)ds}\mathbb{I}_{\{\bar{n}_t \le r_t\}}), \qquad \tilde{U}_*(t) = \mathbb{E}_0(e^{\frac{1}{2}\int_0^t \hat{V}^2(X_s)ds}\mathbb{I}_{\{\bar{n}_t \le r_t\}}). \tag{4.9}$$

Clearly $U_*(t) = \tilde{U}_*(t)$, and by the same Lemma 4.2, we have, as $t \to \infty$,

$$m_1(t) = \langle U(t)\rangle_{P_V} \sim \langle U_*(t)\rangle_{P_V} = \langle \tilde{U}_*(t)\rangle_{P_V} \le \langle \tilde{U}(t, 0)\rangle_{P_V}. \tag{4.10}$$

The operator on the right side of (4.8) has a discrete spectrum, $E_{k,t}$, $k = 1, 2 \dots, |S_t^d|$, where $|S_t^d| = (2r_t + 1)^d$ is the number of points of $S_t^d$, and if $\|\cdot\|$ denotes the norm in $L_2(S_t^d)$, we have (as in [5]) that

$$\langle \tilde{U}(t, 0)\rangle_{P_V} \le \|\tilde{U}(0, \cdot)\| \langle e^{t \max_k\{E_{k,t}\}}\rangle_{P_V} = |S_t^d|^{\frac{1}{2}} \langle e^{t \max_k\{E_{k,t}\}}\rangle_{P_V}. \tag{4.11}$$

As the distribution is continuous, there is a.s. only one site $x_1$ where the highest peak $\xi^{(1)} = \max\{\frac{V^2(x)}{2} : x \in S_t^d\}$ is achieved, and the second highest peak $\xi^{(2)} = \max\{\frac{V^2(x)}{2} : x \in S_t^d, x \ne x_1\}$ is also achieved only at one site (a.s.), denoted $x_2$. Observe that $\max_k\{E_{k,t}\} \le \max_k\{\tilde{E}_{k,t}\}$, where $\tilde{E}_{k,t}$, $k = 1, 2, \dots, |S_t^d|$ are the eigenvalues of the operator

$$\tilde{H} = \kappa\Delta + \xi^{(2)} + \delta_0(x)(\xi^{(1)} - \xi^{(2)}).$$

Moreover $\max_k \tilde{E}_{k,t} = \xi^{(2)} + f_t(\xi^{(1)} - \xi^{(2)})$, where $f_t$ is given in [5] and satisfies the inequality $f_t(a) \le a$. The expected value in (4.11) is then bounded by

$$\begin{aligned}
&\langle e^{t[\xi^{(2)}+f_t(\xi^{(1)}-\xi^{(2)})]}\rangle_{P_V} \\
&\quad = |S_t^d|(|S_t^d|-1)\int_0^\infty dy(F(y))^{|S_t^d|-2}\phi(y)\int_y^\infty dx\phi(x)e^{t[y+f_t(x-y)]} \\
&\quad \le |S_t^d|^2 I(t), \\
&I(t) = \int_0^\infty dy\phi(y)\int_y^\infty dx\phi(x)e^{t[y+f_t(x-y)]},
\end{aligned}\tag{4.12}$$

where $\phi(x) = ce^{-c\frac{x^\alpha}{\alpha}}x^{\alpha-1}$ is the Weibull density and $F(x) = \int_0^x \phi(t)dt$ is its distribution function. If now $\delta(t)$ is a positive function such that $\delta(t) \uparrow \infty$, as $t \to \infty$, using the inequality $f_t(a) \le a$, we find

$$\begin{aligned}
&\int_0^\infty du\phi(u)\int_u^{u+\delta(t)} dv\phi(v)e^{t[u+f_t(v-u)]} \\
&\quad \le \int_0^\infty du\phi(u)\int_u^{u+\delta(t)}\phi(v)e^{tv}dv \\
&\quad \le e^{t\delta(t)}\int_0^\infty \phi(u)\Phi(u)e^{tu}du = \frac{1}{2}e^{t\delta(t)+H_{(\alpha,2c)}(t)},
\end{aligned}\tag{4.13}$$

where $\Phi(x) = e^{-c\frac{x^\alpha}{\alpha}}$ and we take into account that $2\phi(y)\Phi(y) = \phi_2(y)$ is the density of the Weibull distribution with parameters $(\alpha, 2c)$. By Lemma 2.1, for large $t$, $H_{(\alpha,2c)}(t) \sim 2^{-\frac{1}{\alpha-1}}H_{(\alpha,c)}(t)$. Hence, if $\delta(t) = o(v_*(t))$, the integral (4.13) is $o(e^{H(t)-\kappa t})$, for large $t$.

For $v - u \ge \delta(t)$, we use the asymptotics $f_t(a) = a - \kappa + \mathcal{O}(\frac{1}{a})$, as $a \to \infty$. We have

$$\begin{aligned}
&\int_0^\infty du\phi(u)\int_{u+\delta(t)}^\infty \phi(v)e^{t[u+f_t(v-u)]}dv \\
&\quad = e^{-\kappa t+o(t)}\int_0^\infty du\phi(u)\int_{u+\delta(t)}^\infty \phi(v)e^{tv}dv \\
&\quad \le e^{-\kappa t+o(t)}\int_0^\infty du\phi(u)\int_0^\infty \phi(v)e^{tv}dv = e^{H(t)-\kappa t+o(t)}.
\end{aligned}\tag{4.14}$$

By (4.11), (4.12) and (4.14), we see that, for some constant $\bar{K}_d$,

$$\langle \tilde{U}(t,0)\rangle_{P_V} \le \bar{K}_d r_t^{\frac{5d}{2}} I(t) = e^{H(t)-\kappa t+o(t)},$$

and, recalling (4.1) and (4.10), Theorem 1 is proved. □

*Proof of Theorem 2* As for Theorem 1, the proof is based on a lower and an upper estimate, and we will carry it out in detail only for $p = 1$. The extension for $p > 1$, is, as in the previous case, straightforward.

Let $\hat{r}_t := \frac{t}{\ln_+^2 t}$, and set $\tilde{r}_t = \sqrt{d}r_t$, where $r_t$ is as in Lemma 4.2. We shall prove that, as $t \to \infty$, a.s. with respect to the distribution of the potential we have

$$e^{tM_{\hat{r}_t}-\kappa t+o(t)} \le U(t) \le e^{tM_{\tilde{r}_t}-\kappa t+o(t)}. \tag{4.15}$$

By Lemma 2.3, we have a.s., as $t \to \infty$, $M_{\hat{r}_t} = C_d \ln^{\frac{1}{\alpha}} t + o(1)$ and $M_{\tilde{r}_t} - M_{\hat{r}_t} = o(1)$. Hence the inequalities (4.15) imply Theorem 2.

*Lower bound.* We take into account in the expectation (4.7) only the contribution coming from a path which spends "most of the time" at the point $x_*$ (which is unique a.s.) where $\frac{V^2(x)}{2}$ attains the maximum in the sphere with center at the origin and radius $\hat{r}_t$. More precisely, we consider only the contribution of the trajectories which attain the maximal point at some time $\tau \in (0, t)$, and then stay there up to time $t$, so that

$$U(t) \geq P(X_\tau = x_*) e^{(t-\tau)M_{\hat{r}_t}} e^{-\kappa(t-\tau)}. \tag{4.16}$$

For $x \in \mathbb{Z}^d$, let $\ell(x)$ denote the minimal length of a path made of nearest neighbor bonds of $\mathbb{Z}^d$ starting at the origin and ending up at $x$. The probability of attaining $x$ at time $\tau$ along a path of length $\ell(x)$ with a minimal number of jumps is a lower bound:

$$P(X_\tau = x) \geq (\kappa\tau)^{\ell(x)} \frac{e^{-\kappa\tau}}{(2d)^{\ell(x)} \ell(x)!} := q_\tau(x).$$

Clearly $|x| \leq \ell(x) \leq \sqrt{d}|x|$, and it is easy to see that, as $\hat{r}_t \to \infty$, $|x_*| \to \infty$ as well (a.s.). Hence we can apply the Stirling formula, and taking into account that $\ell(x_*) \leq \sqrt{d}\hat{r}_t$, we find for $t$ large enough

$$\begin{aligned} \ln q_\tau(x_*) &= \ell(x_*)\left[\ln \frac{\kappa\tau}{2d} - \ln \ell(x_*) + 1\right] - \kappa\tau - \mathcal{O}(\ln \ell(x_*)) \\ &\geq -\kappa\tau - 2\ell(x_*) \ln \ell(x_*) = -\kappa\tau + \mathcal{O}\left(\frac{t}{\ln t}\right). \end{aligned}$$

Inequality (4.16) now gives $\ln U(t) \geq t M_{\hat{r}_t} - \kappa t + o(t)$, i.e., the lower bound (4.15).

*Upper bound.* As a first step we prove that the asymptotics of $U(t)$, a.e. with respect to the distribution of the potential, is the same as that of the quantity $U_*(t)$ introduced in (4.9). We have

$$\begin{aligned} 0 \leq U(t) - U_*(t) &\leq \mathbb{E}_0(\mathbb{I}_{\{n_t > r_t\}} e^{\frac{1}{2}\int_0^t V^2(X_s)ds}) \\ &= \sum_{r=r_t+1}^{\infty} \mathbb{E}_0(\mathbb{I}_{\{n_t = r\}} e^{\frac{1}{2}\int_0^t V^2(X_s)ds}) \\ &\leq \sum_{r=r_t+1}^{\infty} e^{tM_r} P(n_t = r) = e^{tM_{r_t}} \mathbb{E}(e^{t(M_{n_t} - M_{r_t})}; n_t > r_t). \end{aligned} \tag{4.17}$$

We now use again Lemma 2.3, and evaluate the mean in (4.17) with the help of the Stirling approximation. Observing that $\exp\{ct(\ln^\beta(k + r_t) - \ln^\beta r_t)\}\binom{r_t+k}{k}^{-1} < 1$, for $k \geq 1$, for any $c > 0$ and $\beta \in (0, 1)$, if $t$ is large enough, we get

$$U(t) - U_*(t) \leq e^{tM_{r_t}} \frac{(\kappa t)^{r_t}}{r_t!} = \exp\{tM_{r_t} - \kappa t b(\ln b - 1) + o(\ln t)\}. \tag{4.18}$$

As $M_{r_t} - M_{\hat{r}_t} = o(1)$, a.s., as $t \to \infty$, by our choice of $b$ and the lower bound (4.15), this quantity is $o(U(t))$. In analogy with (4.10) we now have

$$U(t) \sim U_*(t) = \tilde{U}_*(t) \leq \tilde{U}(t, 0), \tag{4.19}$$

and we use spectral analysis for the solution $\tilde{U}(t,0)$ of the problem (4.8), much in the same way as it was done in the proof of Theorem 1.

As a first step we prove a lemma which shows that the high peaks cannot be too close. As set of high peaks we take those points of $S_t^d$ where $\frac{V^2(x)}{2} > B_t := C_d((1-\epsilon)\ln t)^{\frac{1}{\alpha}}$, for some $\epsilon \in (0, \frac{1}{2})$, where $C_d$ is given in Lemma 2.3. Let $\mathcal{P}_t$ be the set of such points.

**Lemma 4.3** *Let $\tilde{\Pi}_t = \{(x_1, x_2) \in \mathcal{P}_t \times \mathcal{P}_t : |x_1 - x_2| < t^\delta\}$, with $\delta + 2\epsilon < 1$. Then, for almost all realizations of the potential, $\tilde{\Pi}_t = \emptyset$, if $t$ is large enough.*

*Proof* The proof is again based on a Borel–Cantelli argument. Let $s = s(t) > 0$ be such that $\limsup_{t\to\infty} \frac{s(t)}{t} = 0$ and consider the set $\Pi_{t,s} = \{(x_1, x_2) \in \mathcal{P}_t \times \mathcal{P}_t : |x_1 - x_2| < (t+s)^\delta\}$. We have

$$P(\Pi_{t,s} \neq \emptyset) \le \sum_{x\in S_t^d} P(V^2(x) > 2B_t)\Bigg(1 - \prod_{\substack{x'\neq x \\ |x-x'|<(t+s)^\delta}} P(V^2(x') \le 2B_t)\Bigg)$$

$$\le \frac{Ct^d}{t^{d(1-\epsilon)}}\left(1 - \left(1 - \frac{1}{t^{d(1-\epsilon)}}\right)^{n_t}\right) \sim \frac{Ct^d n_t}{t^{2d(1-\epsilon)}} \sim \bar{C} t^{d(-1+2\epsilon+\delta)}, \tag{4.20}$$

where $C, \bar{C}$ are constants, $n_t$ is the cardinality of the set $\{x' \neq x : |x - x'| < (t+s)^\delta\}$, and we have taken into account that $\frac{n_t}{t^{d(1-\epsilon)}} \asymp \frac{t^{\delta d}}{t^{d(1-\epsilon)}} \to 0$, as $\epsilon + \delta < 1$. The right side of (4.20) is then summable over a sequence $t_k = k^r$, $k = 1, 2, \ldots$, if we take $r = 1$ for $\beta := d(1 - 2\epsilon - \delta) > 1$ (which is always the case if $d > 1$ and $\epsilon, \delta$ are small enough), and $r$ such that $r\beta > 1$ for $\beta \le 1$. Take moreover $s(t) = (t^{\frac{1}{r}} + 1)^r - t$, so that $t_{k+1} = t_k + s(t_k)$. By the Borel–Cantelli lemma $\Pi_{t_k, s(t_k)} = \emptyset$ almost surely for $k$ large enough, and $\tilde{\Pi}_t \subseteq \Pi_{t_k, s(t_k)}$ for all $t \in [t_k, t_{k+1}]$.

Lemma 4.3 is proved. □

*Conclusion of the proof of Theorem 2 (upper bound).* Consider again the discrete spectrum $E_{k,t}$, $k = 1, 2, \ldots, |S_t^d|$ of the operator on the right of (4.8). For the highest value one finds $\max_k E_{k,t} \le B_t + \max_k \bar{E}_{k,t}$, where, as above, $B_t = C_d((1-\epsilon)\ln t)^{\frac{1}{\alpha}}$, and $\bar{E}_{k,t}$, $k = 1, 2, \ldots, |S_t^d|$, are the eigenvalues of the operator $\bar{H} = \kappa\Delta + \sum_{y\in\mathcal{P}_t} \xi(y)$ on $S_t^d$, with a potential $\xi(y) := \frac{V^2(y)}{2} - B_t$ acting only at the high peaks.

For the spectrum of $\bar{H}$ we use a result from the theory of random Schrödinger operators ([5], Lemma 3.1). As $\max_{y\in S_t^d} \xi(y) \to \infty$ and the minimal distance between the points of $\mathcal{P}_t$ diverges (Lemmas 2.3 and 4.3), there is a function $\delta(t)$, with $\lim_{t\to\infty} \delta(t) = 0$, such that the spectrum of $\bar{H}$ is contained in the interval $[0, \max_{x\in\mathcal{P}_t} \xi(x) - \kappa + \delta(t))$.

Taking into account that $|x| \le \tilde{r}_t$ if $x \in S_t^d$, we have, as $t \to \infty$,

$$\max_k E_{k,t} \le \max_{x\in S_t^d} \frac{V^2(x)}{2} - \kappa + o(1) \le M_{\tilde{r}_t} - \kappa + o(1).$$

Recalling now the relations (4.19), the upper bound in (4.15) is proved. □

*Remark* (*The case of Gaussian variables*) If the variables $V(x)$ have a standard Gaussian distribution, as it is assumed for the stationary problem in [5], there is no asymptotics of the kind stated by Lemma 2.1, as the exponential moment diverges for $t > 1$. One could however study the behavior of the semi-annealed moments $U_p$ and obtain a result analogous to Theorem 2 with $\alpha = 1$.

## 5 Fully Quenched Behavior in a Simplified Model

Although the moment asymptotics of the semi-annealed quasi-stationary Anderson parabolic problem that we consider here is similar to that of the stationary Anderson parabolic problem with potential $\frac{V^2(x)}{2}$, the two models are in fact quite different. The fully quenched behavior of the quasi-stationary problem appears in fact to be closer to that of the non-stationary Anderson parabolic problem with independent white noises at each site. On the whole, the quasi-stationary problem has some features of the stationary problem and some of the non-stationary one, as well as its own features.

It is worth to illustrate this point in a simple, explicitly solvable, model which we briefly discuss in this paragraph. The leading term of the moment asymptotics as $t \to \infty$ is in fact reproduced even by extremely simplified versions.

Take, for instance, the case of constant potential $V(x) = V_0$, for all $x \in \mathbb{Z}^d$. The Feynman–Kac representation (1.6) of the Stratonovich solution of (1.5) is $u(t,x) = e^{V_0 W_t}$. The semi-annealed moment ($p = 1$) is $U(t,x) = \langle u(t,x)\rangle_{P_w} = e^{t\frac{V_0^2}{2}}$. The asymptotics is "almost" the same as for Th. 2 (compare with (4.15)), if $V_0^2/2$ is large, a condition which in the original model is a consequence of the space variation of the potential over a large volume.

Assuming that $V_0^2/2$ is Weibull distributed, the annealed moment is $\langle U(t)\rangle_{P_V} = e^{H(t)}$, and the leading term is the same as for the real model (Th. 1). Observe that $V_0^2/2$ is typically large if the parameter $c$ of the Weibull distribution is small, a condition which may here mimic the absence of space variation.

For the quenched case we have $\frac{\ln u(t,x)}{t} \to 0$, as $t \to \infty$, which is quite unrealistic, because space fluctuations are totally absent.

To take into account space fluctuations we need at least two values of the potential. We reduce the phase space $\mathbb{Z}^d$ to two points, i.e., to the set $S_2 := \{0, 1\}$, with periodic conditions. The Laplacian on $S_2$ is given by the matrix

$$L = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the quasi-stationary Anderson parabolic problem on $S_2$ is

$$\frac{\partial u}{\partial t} = \kappa L u + V(x)\dot{w}_t u, \qquad u(0,x) \equiv 1, \quad t \geq 0,\ x \in S_2, \tag{5.1}$$

which is interpreted as equivalent to an integral equation of the type (1.6). The arguments in Sect. 3 for bounded potentials give existence, uniqueness, and the Feynman–Kac representation (1.7) (where $X_s$ is a random walk with generator $\kappa L$). The semi-annealed moments $U_p(t,x) = \langle (u(t,x))^p\rangle_{P_w}$ are given by the expressions (1.9), where $X_s^{(j)}$, $j = 1, \ldots, p$ are independent random walks on $S_2$. Taking $p = 1$, we see that $U(t,x) := U_1(t,x)$ is a solution of the periodic problem

$$\frac{\partial U(t,x)}{\partial t} = \kappa L U(t,x) + \tilde{V}_x U(t,x), \qquad U(0,x) \equiv 1, \quad t \geq 0,\ x \in S_2, \tag{5.2}$$

where we write for brevity $\tilde{V}_x = V^2(x)/2$. Equation (5.2) is a linear ODE for the vector $\mathbf{U}(t) := (U(t,0), U(t,1))$ of the form $\dot{\mathbf{U}} = A\mathbf{U}$ with matrix

$$A = \begin{pmatrix} \tilde{V}_0 - \kappa & \kappa \\ \kappa & \tilde{V}_1 - \kappa \end{pmatrix}$$

The asymptotic behavior of $\mathbf{U}(t)$ is determined by the leading eigenvalue $\lambda_0$ of $A$:

$$\lambda_0(\tilde{V}_0, \tilde{V}_1) = \frac{\tilde{V}_0 + \tilde{V}_1}{2} - \kappa + \left(\frac{(\tilde{V}_0 - \tilde{V}_1)^2}{4} + \kappa^2\right)^{\frac{1}{2}}. \tag{5.3}$$

As shown by the following results, this simple model reproduces very closely the results of the original one, if $|\tilde{V}_0 - \tilde{V}_1|$ is large. (This condition, which in the original model is due to space variation, can be mimicked, as explained above, by taking $\tilde{V}_i$, $i = 0, 1$ independent Weibull distributed with a small parameter $c$.)

**Proposition 5.1** *If $|\tilde{V}_0 - \tilde{V}_1|/\kappa > \epsilon^{-1}$, with $\epsilon << 1$, then, as $t \to \infty$, the solution of equation (5.2) with initial data $\mathbf{U}(0) = (1, 1)$ has the following asymptotics*

$$U(t, x) \sim \mathbf{U}(0) \cdot \mathbf{e}^{(0)} e_x^{(0)} e^{Mt - \kappa t(1+\mathcal{O}(\epsilon))}, \tag{5.4}$$

*where $M = \max\{\tilde{V}_0, \tilde{V}_1\}$, $e_x^{(0)}$, $x = 0, 1$, are the components of the normalized eigenvector $\mathbf{e}^{(0)}(\tilde{V}_1, \tilde{V}_2)$ corresponding to the eigenvalue $\lambda_0$, and $\cdot$ denotes the scalar product.*

*Proof* As $\mathbf{U}(0) \cdot \mathbf{e}^{(0)} \neq 0$, $\mathbf{U}(t) \sim \mathbf{U}(0) \cdot \mathbf{e}^{(0)} e^{\lambda_0 t} \mathbf{e}^{(0)}$, for large $t$. The proof then follows by expanding the expression for $\lambda_0$ in (5.3) for small $\frac{\kappa}{|\tilde{V}_0 - \tilde{V}_1|}$. □

**Proposition 5.2** *If $\tilde{V}_0$, $\tilde{V}_1$ are independent and $(\alpha, c)$-Weibull, then, for large $t$*

$$\langle U(t, x)\rangle_{P_V} = e^{H_{(\alpha,c)}(t) - \kappa t + o(t)}, \quad x \in S_2. \tag{5.5}$$

*Proof* The lower bound $\langle U(t, x)\rangle_{P_V} \geq e^{-\kappa t + H_{(\alpha,c)}(t)}$ is obtained in the same way as in the proof of Lemma 4.1.

For the upper bound we use, as for the semi-annealed asymptotics (5.4), the large $t$ asymptotics of $\mathbf{U}(t)$. We take $\delta(t)$ as in (4.13), and the proof follows the lines of (4.13), (4.14). As $\lambda_0(u, v) \leq \max\{u, v\}$, and $|\mathbf{U}(0) \cdot \mathbf{e}^{(0)}| \leq 2$, we find

$$\langle U(t, x)\mathbb{I}_{\{|\tilde{V}_0 - \tilde{V}_1| \leq \delta(t)\}}\rangle_{P_V} \leq 2\int_0^\infty \phi(u)du \int_u^{u+\delta(t)} \phi(v)e^{tv}dv = o(e^{H_{(\alpha,c)}(t) - \kappa t}).$$

If $|\tilde{V}_0 - \tilde{V}_1| > \delta(t)$ we have, as $t \to \infty$, $\lambda_0(u, v) = \max\{u, v\} - \kappa + o(1)$, so that

$$\langle U(t, x)\mathbb{I}_{\{|\tilde{V}_0 - \tilde{V}_1| > \delta(t)\}}\rangle_{P_V} \leq e^{-\kappa t + o(t)} \int_0^\infty \phi(u)du \int_u^\infty \phi(v)e^{tv}dv = e^{H(t) - \kappa t + o(t)}. \quad \square$$

We proceed to study the fully quenched behavior for this model. Consider the solution of (5.1) $u(t, x) = \mathbb{E}_x e^{\int_0^t V(X_{t-s})dw_s}$, and set $u_x(t) = u(t, x)$, $V_x = V(x)$, $x = 0, 1$. From the expression of the increment of $u_0$

$$u_0(t + \Delta t) = (1 - \kappa\Delta t)e^{V_0 \Delta w} u_0(t) + \kappa u_1(t)\Delta t + o(\Delta_t)$$

and a similar one for $u_1$, we get the stochastic differential equation

$$\begin{aligned} du_0 &= V_0 u_0 dw + \left(\frac{V_0^2}{2} - \kappa\right) u_0 dt + \kappa u_1 dt, \\ du_1 &= V_1 u_1 dw + \left(\frac{V_1^2}{2} - \kappa\right) u_1 dt + \kappa u_0 dt. \end{aligned} \tag{5.6}$$

In analogy with the phase-amplitude formalism for the 1-d random Schrödinger operator, we set $v(t) = u_0(t) + u_1(t)$, $p(t) = \frac{u_0(t)}{v(t)}$, $q(t) = \frac{u_1(t)}{v(t)} = 1 - p(t)$. By adding equations (5.6) we find

$$dv = v\left[(V_0 p + V_1 q)dw + \frac{V_0^2 p + V_1^2 q}{2} dt\right]. \tag{5.7}$$

The Ito formula $dF(u, v) = F_u du + F_v dv + \frac{1}{2} F_{uu} du^2 + \frac{1}{2} F_{vv} dv^2 + F_{uv} du dv$ gives $dp = d(u_0/v) = du_0/v - u_0 dv/v^2 - du_0 dv/v^2 + u_0(dv)^2/v^3$, and, after some computations we get

$$dp = pq(V_0 - V_1)dw - \frac{p-q}{2}[(V_0 - V_1)^2 pq + 2\kappa]dt. \tag{5.8}$$

Equation (5.8) shows that the diffusion process $p(t)$ in the interval [0, 1] cannot reach the edges ($pq = 0$) and has a unique invariant measure $\pi$. From (5.7), setting $\bar{V}(s) = V_0 p(s) + V_1 q(s)$, we get

$$\begin{aligned} \ln v(t) &= \int_0^t \bar{V}(s) dw_s + \frac{1}{2} \int_0^t [V_0^2 p(s) + V_1^2 q(s)] ds - \frac{1}{2} \int_0^t \bar{V}^2(s) ds \\ &= \int_0^t \bar{V}(s) dw_s + \frac{1}{2} \int_0^t [(V_0 - \bar{V}(s))^2 p(s) + (V_1 - \bar{V}(s))^2 q(s)] ds. \end{aligned} \tag{5.9}$$

An immediate consequence of formula (5.9) is the following result.

**Proposition 5.3** *As $t \to \infty$, the following limit holds, $P_w$-a.s.,*

$$\lim_{t\to\infty} \frac{\ln v(t)}{t} = \gamma := \frac{1}{2} \langle (V_0 - \bar{V})^2 p + (V_1 - \bar{V})^2 q \rangle_\pi, \tag{5.10}$$

*where $\pi$ is the invariant measure for the diffusion process $p(t)$ given by* (5.8).

Equation (5.8) shows that if $|V_0 - V_1|$ is large there is a drift to the point $p = 1/2$, which is large unless $p \approx q \approx 1/2$, and vanishes for $p = 1/2$. Hence $\pi$ is concentrated near the point $p = q = 1/2$ and $\gamma \approx (V_0 - V_1)^2/8$. On the basis of this result we can formulate a conjecture on the fully quenched behavior of the original model.

Let us assume that the variables $V(x)$ have a symmetric Weibull distribution, i.e., their distribution is symmetric and such that $V^2(x)/2$ are $(\alpha, c)$-Weibull variables. For the leading term of the fully quenched asymptotics of the model on $\mathbb{Z}^d$, we can then formulate, as a conjecture, the following result.

**Theorem 3** *As $t \to \infty$, the solution of the quasi-stationary Anderson Parabolic Problem* (3.2) *has the following asymptotics, $P_w \times P_V$-a.e.*

$$\ln u(t, x) \sim t \max_{\substack{|x-x'|=1 \\ |x|,|x'|\le\kappa t}} \frac{(V(x) - V(x'))^2}{8} = \widehat{C}_d \ln^{\frac{1}{\alpha}} t (1 + o(1)), \tag{5.11}$$

*where $\widehat{C}_d = 2^{-\frac{1}{\alpha}} C_d$.*

About the proof of this statement, which, as we said, will be the object of future work, we may say that the lower estimate is given by a simple generalization of the argument for the two-point model.

The upper estimate is, as usual, harder. The analysis of the next term is also important, as it gives relevant information on the quenched behavior of $u(t,x)$.

*Remark* The value of $\widehat{C}_d$ is obtained by an analogue of Lemma 2.3. In fact, as the density of $V_i$, $i=0,1$ is $\varphi(x)=2^{-\alpha}c|x|^{2\alpha-1}e^{-c\frac{x^{2\alpha}}{2^\alpha\alpha}}$, $x\in\mathbb{R}$, one finds for large $a>0$

$$\ln P\left(\left\{\frac{(V_1-V_2)^2}{2}>a\right\}\right)\sim -\frac{ca^\alpha}{2^{2\alpha-1}\alpha}.$$

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